EFFICIENT ESTIMATION OF ONE-DIMENSIONAL DIFFUSION FIRST PASSAGE TIME DENSITIES VIA MONTE CARLO SIMULATION

TOMOYUKI ICHIBA AND CONSTANTINOS KARDARAS

ABSTRACT. We propose a method for estimating first passage time densities of one-dimensional diffusions via Monte Carlo simulation. Our approach involves a representation of the first passage time density as expectation of a functional of the three-dimensional Brownian bridge. As the latter process can be simulated exactly, our method leads to almost unbiased estimators. Furthermore, since the density is estimated directly, a convergence of order $1/\sqrt{N}$, where N is the sample size, is achieved, the last being in sharp contrast to the slower non-parametric rates achieved by kernel smoothing of cumulative distribution functions.

0. INTRODUCTION

The problem of computing the distribution of the first time that a diffusion crosses a certain level naturally arises in many different contexts. As probably the most prevalent we mention quantitative finance, where first passage times are used in credit risk (times of default) as well as in defining exotic contingent claims (so-called barrier options). In this paper, we focus on the numerical computation of the probability density function of first-passage times associated with a general one-dimensional diffusion.

Densities of first passage times have known analytic expressions only in very particular cases. The primary example is Brownian motion with certain (constant) drift and diffusion rates, where one uses a combination of Girsanov's theorem and the special case of standard (driftless) Brownian motion. The first passage time density for the latter case can be obtained by the reflexion principle — see, for example, [11, 2.6A]. Another example of explicit form of the first-passage time density is the case of the radial Ornstein-Uhlenbeck process — see [7] and [10].

In absence of general analytic expressions for first passage time distributions, computational methods are indispensable and, in fact, widely used. One branch uses Volterra integral equations — we mention [5], [16] and [2] as representative papers dealing with this approach. Alternatively, one can use Monte-Carlo simulation to attack this problem. The simplest scheme uses the so-called Euler scheme in order to approximate the solution of the stochastic differential equation governing the diffusion at predetermined grid time-points ih, $i = 0, 1, \ldots$, where h is the step size, stops at

the first time that the diffusion crosses the level of interest, and continuing this way obtain an estimator for the cumulative distribution function of the first-passage time. As the Euler scheme is only approximate[1], it causes bias in the estimation of the probability distribution function. (See [8] for general discussion, [9] for the evaluation of error via partial differential equations and [3] for a sharp large deviation principle approach.) The issue with the bias becomes immensely more severe in the numerical computation of the density, as it will involve some kind of numerical differentiation of the (non-smooth) empirical distribution function. Even if one uses an exact simulation approach for the diffusion in question (which is, of course, available only in special cases), the estimator for the density will have huge variance. To top it all, even if all the aforementioned problems can be eliminated, one can never hope for convergence of the estimators to the true density in order $1/\sqrt{N}$, where N is the "path-sample" size, as the problem is non-parametric.

In this work, we offer an alternative approach which has clear advantages. First, we arrive at a representation of the density function in terms of expectation of a functional of a three-dimensional Brownian bridge. This makes it possible to estimate *directly* the first passage time density without having to rely on estimators of the cumulative distribution function, achieving this way the "parametric" rate of convergence $1/\sqrt{N}$, where N is the sample size. Furthermore, only the three-dimensional Brownian bridge is involved in the simulation, which can be carried out exactly. There is an integral involving the previous three-dimensional Brownian bridge, which can be approximated via a Riemann sum; therefore, the error of the approximation can be estimated efficiently. By construction, our method significantly improves both the bias and variance of the density estimation obtained via the empirical distribution function. The only potential problem of our approach is large-time density estimation, since the thin grid that has to be used in the simulation of the Brownian bridge will result in high computational effort. To circumvent this issue, we notice that the tails of the first-passage distribution usually decrease exponentially with a rate that can be expressed as the principal eigenvalue of a certain Dirichlet boundary problem involving a second-order ordinary differential equation. This implies that a mixture of Monte-Carlo and ordinary differential equation techniques can be efficiently utilized and improve the quality of our estimator.

The structure of the paper is simple. In Section 1, the problem is formulated and the key representation formula is obtained. In Section 2, we discuss the Monte-Carlo estimator of the first passage time density function, and study its large-sample properties. In Section 3, the relation between the exponential tail decay of the probability density and the eigenvalues of a Dirichlet

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[1]To account for the fact the passage can potentially happen in-between the sampled points, a Brownian bridge interpolation may be used — this means that the conditional probability distribution of first-passage time occurred in the interval given the simulated values at the end points is approximated by that of a Brownian bridge. This could potentially reduce the bias, but the whole scheme is still only approximate and some bias remains.

boundary problem is discussed. The proofs of all the results are deferred to Appendix A in order to keep the presentation smooth in the main body of the paper.

1. A REPRESENTATION OF FIRST-PASSAGE-TIME DENSITIES

1.1. **The set-up.** Consider a one-dimensional diffusion X with dynamics

$$(1.1) \qquad \mathrm{d}X_t = a(X_t)\mathrm{d}t + \mathrm{d}W_t, \quad t \in \mathbb{R}_+.$$

Above, W is a standard one-dimensional Brownian motion. The restrictions on the drift function a that we shall impose later (Assumptions 1.2) ensure that (1.1) has a weak solution, unique in the sense of probability law, for any initial condition $x \in (0, \infty)$. Let $\mathbb{P}_x^a$ will denote the law on the canonical path-space $C(\mathbb{R}_+; \mathbb{R})$ of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ that makes the coordinate processes behave according to (1.2) and is such that $\mathbb{P}_x^a[X_0 = x] = 1$.

Define $\tau_0 := \inf\{t \in \mathbb{R}_+ \mid X_t = 0\}$ to be the first passage time of X at level zero. We shall consider the problem of finding convenient, in terms of numerical approximation using the Monte-Carlo simulation technique, representations of the quantity

$$p_x^a(t) := \frac{\partial}{\partial t}\mathbb{P}_x^a[\tau_0 \leq t]; \quad x \in (0, \infty),\ t \in \mathbb{R}_+,$$

i.e., the density of the first-passage time of the diffusion at level zero.

Remark 1.1. The fact that we are using unit diffusion coefficient in (1.1) by no means entails loss of generality in our discussion. Indeed, consider a general one-dimensional diffusion Y with dynamics

$$(1.2) \qquad \mathrm{d}Y_t = b(Y_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}W_t, \quad t \in \mathbb{R}_+,$$

where W is a standard one-dimensional Brownian motion, such that $Y_0 = y \in \mathbb{R}$. If (1.2) has a weak solution unique in the sense of probability law, we may assume without loss of generality that $\sigma \geq 0$. (Indeed, otherwise we replace σ by $|\sigma|$ in (1.2) and we obtain the same law for the process Y.) Consider a level $\ell < y$. Under the mild assumption that $1/\sigma$ is locally integrable, the transformation $X = \int_\ell^Y (1/\sigma(z))\,\mathrm{d}z$ defines a diffusion with dynamics $\mathrm{d}X_t = a(X_t)\mathrm{d}t + \mathrm{d}W_t$, for a function a that is easily computable from b and σ. With $x := \int_\ell^y (1/\sigma(z))\mathrm{d}z$, the first passage time of Y with $Y_0 = y$ at level ℓ is equal to the first passage time of X with $X_0 = x$ at level zero.

1.2. **The representation.** The following assumption on the drift function in (1.1) will allow us to arrive at a very convenient representation for the density function p_x^a.

Assumption 1.2. The function a restricted on $[0, \infty)$ is continuously differentiable, and satisfies

$$\int_0^\infty \exp\left(-2\int_0^w a(z)\mathrm{d}z\right)\mathrm{d}w = \infty.$$

In particular, under Assumption 1.2, a is locally square integrable on $[0, \infty)$ and the function

$$(1.3) \qquad \gamma := \frac{a^2 + a'}{2}.$$

is continuous and locally integrable. The theory of one-dimensional diffusions ensures that for all $x \in (0, \infty)$ there exists a probability $\mathbb{P}_x^a$ on $C(\mathbb{R}_+, \mathbb{R})$ such that the coordinate process X has dynamics given by (1.1). Assumption 1.2 also ensures that X^{τ_0}, which is X stopped at level zero, does not explode to infinity — see, for example, [11, Proposition 5.32 (iii)].

Proposition 1.3. *Suppose that Assumption 1.2 is in force. On $C([0, 1]; \mathbb{R}^3)$, consider the probability $\mathbb{P}_{\mathsf{BB}^3}$ under which the coordinate process β is a standard 3-dimensional Brownian bridge. Then,*

$$p_x^a(t) = q_x(t) \exp\left(-\int_0^x a(v) \mathrm{d}v\right) \mathbb{E}_{\mathsf{BB}^3}\left[\exp\left(-t \int_0^1 \gamma\left(\left|u x \mathrm{e}_1 + \sqrt{t}\beta_u\right|\right) \mathrm{d}u\right)\right]$$

holds for all $t \in \mathbb{R}_+$, where $\mathrm{e}_1 := (1, 0, 0)^\top$ and q_x is the density given for all $t \in \mathbb{R}_+$ by

$$(1.4) \qquad q_x(t) \equiv p_x^0(t) = \frac{x}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right),$$

corresponding to the first-passage time to zero of a standard Brownian motion starting from x.

2. Monte-Carlo Density Estimation

We now discuss issues related to estimation of the density p_x^a. For the purposes of this and the next section, we *fix* a drift function a satisfying Assumption 1.2 and we write p_x for p_x^a in order to simplify notation.

2.1. **Convergence.** It is clear how to get an estimate of the density $p_x(t)$ for a given $t \in \mathbb{R}_+$, at least in theory. One simulates N independent paths of 3-dimensional Brownian bridge $\widehat{\beta}^1, \ldots, \widehat{\beta}^N$, and then defines the estimator $\widehat{p}_x^N(t)$ for $p_x(t)$ via

$$(2.1) \qquad \widehat{p}_x^N(t) := q_x(t) \exp\left(-\int_0^x a(v) \mathrm{d}v\right) \frac{1}{N} \sum_{i=1}^N \exp\left(-t \int_0^1 \gamma\left(\left|u x \mathrm{e}_1 + \sqrt{t}\widehat{\beta}_u^i\right|\right) \mathrm{d}u\right),$$

where recall that q_x is given in (1.4). By the strong law of large numbers, the estimator $\widehat{p}_x^N(t)$ converges almost surely to the true density $p_x(t)$ as N goes to infinity for each fixed $t \in \mathbb{R}_+$. Moreover, the estimator $\widehat{p}_x^N(t)$ is unbiased[2] and the variance of the estimator $\widehat{p}_x^N(t)$ decreases in the order of $1/N$, for every fixed $t \in \mathbb{R}_+$, as a direct consequence of (2.1). In order to get weak convergence of the whole empirical densities $\left(\widehat{p}_x^N(t)\right)_{t \in \mathbb{R}_+}$, as well as the uniform rate of convergence over compact time-intervals, we introduce an additional assumption.

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[2]Of course, $\mathbb{E}_{\mathsf{BB}^3}\left[\widehat{p}_x^N(t)\right] = p_x(t)$ only holds if we assume that we actually have the whole path of each Brownian bridge simulated exactly, which is not possible in practice. However, we can simulate exactly discretized paths of the Brownian bridge, and then can easily estimate the order of bias from the Riemann approximation of the integral. In this respect, see also §3.2.

Assumption 2.1. Together with Assumption 1.2, we ask that the function γ of (1.3) is such that $\inf_{z \in \mathbb{R}_+} \gamma(z) > -\infty$, as well as that a' is locally Lipschitz continuous on $\mathbb{R}_+$ with Lipschitz constant growing at most polynomial rate, that is, there exist constants $c_1 > 0$, $c_2 > 0$, $n \in \mathbb{N}$, such that

$$(2.2) \qquad \sup_{0 \le v_1 < v_2 \le \kappa} \left| \frac{a'(v_2) - a'(v_1)}{v_2 - v_1} \right| \le (c_1 + c_2 \kappa^n) \text{ holds for all } \kappa > 0.$$

The next two results are concerned with a central limit theorem for the whole density function estimator as well as the uniform rate of convergence on compact intervals of $\mathbb{R}_+$.

Proposition 2.2. *Suppose that Assumption 2.1 holds. For $N \in \mathbb{N}$, define $\eta^N := \sqrt{N} \left(\widehat{p}_x^N - p_x \right)$. Then, the family of stochastic processes $\{\eta^N \mid N \in \mathbb{N}\}$ is tight. As $N \to \infty$, η^N converges weakly to a centered Gaussian process with continuous covariance function Γ, where, with $I(t) = \int_0^1 \gamma \left(|ux\mathrm{e}_1 + \sqrt{t}\beta_u| \right) \mathrm{d}u$ for $t \in \mathbb{R}_+$,*

$$\Gamma(s,t) = p_x(s)p_x(t) \exp\left(-2 \int_0^x a(v)\mathrm{d}v \right) \mathrm{cov}_{\mathsf{BB}^3}[\exp(-sI(s)), \exp(-tI(t))], \quad (s,t) \in \mathbb{R}_+^2.$$

Proposition 2.3. *Under Assumption 2.1, for any fixed $T \in \mathbb{R}_+$ the sequence*

$$\left(\sqrt{N} \max_{t \in [0,T]} \left| \widehat{p}_x^N(t) - p_x(t) \right| \right)_{N \in \mathbb{N}}$$

is bounded in probability.

Remark 2.4. In a similar manner, for fixed x_1 and x_2 in $(0, \infty)$ with $x_1 < x_2$, we may show that

$$\left(\sqrt{N} \max_{x \in [x_1, x_2]} \left| \widehat{p}_x^N(t) - p_x(t) \right| \right)_{N \in \mathbb{N}}$$

is bounded in probability. Moreover, under some additional conditions on differentiability of a, we may estimate the partial derivatives of $p_x(t)$ with respect to (x,t) via differentiating the estimator with respect to the variable of interest.

3. The Rate Function

Recall that we are dropping the qualifying "a" from "p_x^a" in order to simplify notation. Define implicitly the function λ_x via

$$p_x(t) = q_x(t) \exp\left(-\int_0^x a(v)\mathrm{d}v \right) \exp\left(-t\lambda_x(t) \right), \quad t \in \mathbb{R}_+.$$

In other words, and in view of Proposition 1.3, we have

$$(3.1) \qquad \lambda_x(t) := -\frac{1}{t} \log \left(\mathbb{E}_{\mathsf{BB}^3} \left[\exp\left(-t \int_0^1 \gamma \left(|ux\mathrm{e}_1 + \sqrt{t}\beta_u| \right) \mathrm{d}u \right) \right] \right), \text{ for } t \in \mathbb{R}_+.$$

3.1. **Theoretical results.** Proposition 2.3 ensures the uniform convergence of the estimator on finite intervals $[0, T]$ for fixed $T \in (0, \infty)$. Of course, it is almost always the case that $\lim_{t \to \infty} p_x(t) = 0$. For large $t \in \mathbb{R}_+$, $\lambda_x(t)$ gives a better understanding of the behavior of the density function, as it represents in a certain sense the exponential decrease of $p_x(t)$; therefore, it makes more sense to focus on λ_x rather than p_x. In fact, the following result implies that the function λ_x is frequently bounded on $\mathbb{R}_+$ — this is the case, for example, when γ is bounded from below.

Proposition 3.1. *Let Assumption 1.2 be valid. Then,*

$$(3.2) \qquad \inf_{z \in \mathbb{R}_+} \gamma(z) \le \inf_{t \in \mathbb{R}_+} \lambda_x(t) \le \limsup_{t \to \infty} \lambda_x(t) \le \inf_{\kappa > 0} \left\{ m(\kappa + x) - \frac{\pi^2}{2\kappa^2} \right\},$$

where $m(w) = \max_{0 \le z \le w} \gamma(z)$ for $w > 0$. Furthermore, if γ is bounded from below, then

$$(3.3) \qquad \lim_{t \downarrow 0} \lambda_x(t) = \frac{1}{x} \int_0^x \gamma(u) \mathrm{d}u = \int_0^1 \gamma(ux) \mathrm{d}u.$$

Remark 3.2. The inequalities (3.2) only imply bounds for the inferior and superior limit of $\lambda_x(t)$ as $t \to \infty$. In fact, it is expected that $\lim_{t \to \infty} \lambda_x(t)$ exists, possibly except in pathological cases. Let us argue below for this point on a rather loose and intuitive level.

The stopping time τ_0 can be approximated by the sequence $(\tau_0^n)_{n \in \mathbb{N}}$, where, for all $n \in \mathbb{N}$, $\tau_0^n := \inf\{t \ge 0 \mid X_t \notin (0, n)\}$ is the first exit time of X from the interval $(0, n)$. Therefore, the rate function λ_x may be approximated by the corresponding rate functions corresponding to τ_0^n, $n \in \mathbb{N}$. It follows from [6] that the density function p_x^n of τ_0^n has the eigenvalue expansion (see also [12], [17] for similar problems)

$$p_x^n(t) = \frac{\partial}{\partial t} \mathbb{P}_x^a [\tau_0^n \le t] = \sum_{k=1}^{\infty} e^{-\mu_k^n t} \psi_k^n(x), \text{ for } (t, x) \in \mathbb{R}_+ \times (0, n), \ n \in \mathbb{N},$$

for some functions $\{\psi_k^n \mid k \in \mathbb{N}\}$ computed from the eigenfunctions $\{\varphi_k^n \mid k \in \mathbb{N}\}$ and the corresponding eigenvalues $0 < \mu_1^n < \mu_2^n < \dots$ of the Dirichlet problem

$$(3.4) \qquad \frac{1}{2} \varphi''(z) + a(z) \varphi'(z) = -\mu \, \varphi(z) \quad \text{for } z \in (0, n),$$

for $\varphi \in C^{2,\alpha}([0, n], \mathbb{R}_+)$ with $\lim_{z \to 0} \varphi(z) = 0 = \lim_{z \to n} \varphi(z)$. Thus, the limit as $t \to \infty$ for the rate function of τ_0^n is exactly the principal eigenvalue μ_1^n:

$$\lim_{t \to \infty} \lambda_x^n(t) = - \lim_{t \to \infty} \frac{1}{t} \log \left(\frac{p_x^n(t)}{q_x(t)} \right) = \mu_1^n \text{ for all } n \in \mathbb{N},$$

which does not depend on the initial value $x > 0$. Since $\tau_0 = \lim_{n \to \infty} \tau_{0,n}$, it is conjectured that the limit of $\lambda_x(t)$ as $t \to \infty$ actually exists and is equal to $\lim_{n \to \infty} \mu_1^n$. A thorough study of finding a reasonable sufficient conditions for

$$\lim_{t \to \infty} \lambda_x(t) = \lim_{n \to \infty} \mu_1^n = \lim_{n \to \infty} \lim_{t \to \infty} \lambda_x^n(t)$$

to hold for $x > 0$ lies beyond the scope of this paper.

(a) Density functions p_x and $\widehat{p}_x^N$.

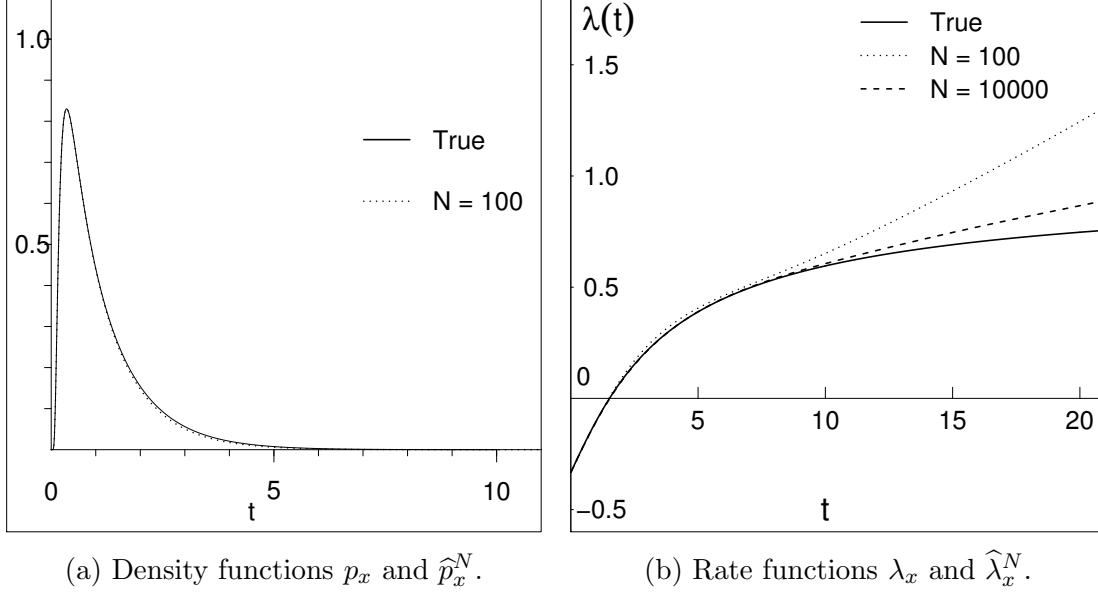(b) Rate functions λ_x and $\widehat{\lambda}_x^N$.

FIGURE 1. Comparisons of the simulated density and rate functions in (2.1) and (3.6) with theoretical values for the OU process with $a(z) = -z$ ($z \in \mathbb{R}_+$) and $x = 1$.

3.2. **Practical issues.** In view of Proposition 2.2 and Proposition 2.3, the estimator $(\widehat{p}_x^N(t))_{t \in \mathbb{R}_+}$ of (2.1) convergences uniformly with rate $1/\sqrt{N}$ over compact time-intervals. In practice, the computation of (2.1) is implemented by generating a standard 3-dimensional Brownian bridge, which is simulated in an exact way over a thin enough grid. The approximation error for the Riemann integral over the *finite* interval $[0,1]$ in (2.1) can be controlled very efficiently. More precisely, the numerical computation of the exponential functional of the Brownian bridge in (2.1) can be carried out using the fourth-order Runge-Kutta scheme which is proposed and analyzed in [14]. Under appropriate mild regularity conditions on the function γ, it is shown that this numerical scheme is weak order of convergence 4. For this numerical issue, consult the original paper [14] and the related monographs [8], [13] and [15].

A potential problem with our estimator $(\widehat{p}_x^N(t))_{t \in \mathbb{R}_+}$ can arise for *large t*, that is, the density function at the tail. Note that what is meant here is that the *relative* error of the estimator of $p_x(t)$ tends to be large; the absolute error tends to be extremely small, as $p_x(t)$ is very close to being zero for large $t \in \mathbb{R}_+$. To visualize the issue, it is helpful to study by experiment the large-time behavior of the rate function (3.1) when X is an Ornstein-Uhlenbeck (OU) process starting with $x = 1$. Here $a(z) = -z$ and $\gamma(z) = (z^2 - 1)/2$ for $z \in \mathbb{R}_+$. The first-passage time density of this OU process is known analytically (see, for example, [10, equation (8)]) and reads

$$(3.5) \qquad p_1(t) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sinh^{3/2}(t)} \exp\left(\frac{1 + t - \coth(t)}{2}\right), \text{ for } t \in \mathbb{R}_+.$$

The true density (3.5) and the estimated density (2.1) with $N = 100$ simulations are shown over interval $[0, 10]$ in Figure 1(a). Note that even with this small number of simulations, the two curves are almost indistinguishable. Figure 1(b) contains graphs of the more refined rate functions. In this scale, one can see that the estimation of the exponential rate of decay of the density for large t is not as good. (As can be seen from 1(b), on the interval $[0, 7]$, the true and estimated rate functions almost coincide. However, on the interval $[7, 20]$, the estimated rate functions $\widehat{\lambda}_x^N$ are much larger than the true rate function λ_x, which implies that the estimator (2.1) underestimates the tail probability.) In fact, the estimated asymptotic rate seems to increase linearly instead of converging to a finite limit. This becomes clear once one notes that, in this OU example,

$$-t \int_0^1 \gamma\left(\left|ux\mathrm{e}_1 + \sqrt{t}\beta_u\right|\right)\mathrm{d}u = t^2 \frac{1}{2}\int_0^1 |\beta_u|^2 \mathrm{d}u + t^{3/2}x\int_0^1 \langle \mathrm{e}_1, \beta_u\rangle\, \mathrm{d}u + t\frac{x^2}{6};$$

therefore, the estimator for λ_x for sample size $N \in \mathbb{N}$ becomes

$$(3.6) \qquad \widehat{\lambda}_x^N(t) = \frac{x^2}{6} - \frac{1}{t}\log\left(\frac{1}{N}\sum_{i=1}^N \exp\left(t^2\frac{1}{2}\int_0^1 |\widehat{\beta}_u^i|^2 \mathrm{d}u + t^{3/2}x\int_0^1 \left\langle \mathrm{e}_1, \widehat{\beta}_u^i\right\rangle \mathrm{d}u\right)\right).$$

Observe that the leading term in the estimator of λ_x will be increasing linearly in t.

In order to overcome this poor situation in the tail, one can use a mixture method combining the estimator (2.1) on the finite interval $[0, T]$ and an estimator for the tail probability of the form

$$c_* q_x(t)\exp(-\lambda t), \text{ for } t \in [T, \infty),$$

for some choice of large threshold T, where λ is the principal eigenvalue of the Dirichlet problem (3.4) for some choice of large threshold n, and c_* is chosen so that the density estimator is continuous. The principal eigenvalue can be numerically computed from the Sturm-Liouville problem

$$-\left[\exp\left(2\int_0^z a(u)\mathrm{d}u\right)\varphi'(z)\right]' = 2\lambda\exp\left(2\int_0^z a(u)\mathrm{d}u\right)\varphi(z) \quad \text{for } z \in [0, n]$$

(see [20]) with the same Dirichlet boundary conditions, by use of either the variational method or the Liouville transform method. In the variational method, the principal eigenvalue is obtained by minimizing numerically the corresponding Rayleigh quotient. The Liouville transform method turns the Sturm-Liouville equation into a Schrödinger-type equation, which is then numerically solvable with discrete approximation. Both numerical methods are well studied — see [1] and their references within.

Appendix A. Proofs

A.1. **Proof of Proposition 1.3.** Consider the non-negative $\mathbb{P}_x^a$-supermartingale

$$(A.1) \qquad\qquad Z := \exp\left(-\int_0^{\cdot\wedge\tau_0} a(X_u)\mathrm{d}W_u - \frac{1}{2}\int_0^{\cdot\wedge\tau_0} a^2(X_u)\mathrm{d}u\right).$$

As follows from a modification of [11, Exercise 5.5.38] for the restricted state space $[0, \infty)$, Z is a $\mathbb{P}_x^a$-martingale. With $\mathbb{Q}_x$ being the probability on $C(\mathbb{R}_+; \mathbb{R})$ that makes the coordinate process X behave like a Brownian motion staring from x and stopped when it reaches level zero, Girsanov's theorem implies that

$$(A.2) \qquad \frac{\mathrm{d}\mathbb{Q}_x}{\mathrm{d}\mathbb{P}_x^a}\bigg|_{\mathcal{F}_t} = Z_t.$$

for all $t \in \mathbb{R}_+$. Moreover, since $X_{\tau_0} = 0$, Itô's formula (under $\mathbb{Q}_x$) implies that, on the set $\{\tau_0 < \infty\}$,

$$-\int_0^x a(v)\mathrm{d}v = \int_x^{X_{\tau_0}} a(v)\mathrm{d}v = \int_0^{\tau_0} a(X_u)\mathrm{d}X_u + \frac{1}{2}\int_0^{\tau_0} a'(X_u)\mathrm{d}u.$$

Combining this with (1.1) and (1.3), the stochastic exponential defined in (A.1) under $\mathbb{Q}_x$ satisfies

$$(A.3) \qquad \begin{aligned} \frac{1}{Z_{\tau_0}} &= \exp\bigg(\int_0^{\tau_0} a(X_u)[\mathrm{d}X_u - a(X_u)\mathrm{d}u] + \frac{1}{2}\int_0^{\tau_0} a^2(X_u)\mathrm{d}u\bigg) \\ &= \exp\bigg(\int_0^{\tau_0} a(X_u)\mathrm{d}W_u^{\mathbb{Q}_x} + \frac{1}{2}\int_0^{\tau_0} a^2(X_u)\mathrm{d}u\bigg) \\ &= \exp\bigg(-\int_0^x a(v)\mathrm{d}v - \frac{1}{2}\int_0^{\tau_0} a'(X_u)\mathrm{d}u - \frac{1}{2}\int_0^{\tau_0} a^2(X_u)\mathrm{d}u\bigg) \\ &= \exp\bigg(-\int_0^x a(v)\mathrm{d}v - \int_0^{\tau_0} \gamma(X_u)\mathrm{d}u\bigg), \end{aligned}$$

on $\{\tau_0 < \infty\}$, where $W^{\mathbb{Q}_x}$ is a standard Brownian motion under $\mathbb{Q}_x$. Therefore, (A.2) and (A.3) imply

$$\mathbb{P}_x^a[\tau_0 \leq t] = \mathbb{E}_{\mathbb{Q}_x}\bigg[\frac{1}{Z_{t \wedge \tau_0}}\mathbb{I}_{\{\tau_0 \leq t\}}\bigg] = \mathbb{E}_{\mathbb{Q}_x}\bigg[\exp\bigg(-\int_0^x a(v)\mathrm{d}v - \int_0^{\tau_0}\gamma(X_u)\mathrm{d}u\bigg)\mathbb{I}_{\{\tau_0 \leq t\}}\bigg]$$

for $(t, x) \in (\mathbb{R}_+)^2$.

Note that the density function of τ_0 under $\mathbb{Q}_x$ is given by q_x in (1.4). Using the regular conditional $\mathbb{Q}_x$-expectation of $\exp(\int_0^{\tau_0} \gamma(X_u)\mathrm{d}u)$, given $\tau_0 = t$, we can write

$$(A.4) \qquad p_x^a(t) = \mathbb{P}_x^a[\tau_0 \in \mathrm{d}t] = q_x(t)\exp\bigg(-\int_0^x a(v)\mathrm{d}v\bigg)\mathbb{E}_{\mathbb{Q}_x}\bigg[\exp\bigg(-\int_0^{\tau_0}\gamma(X_u)\mathrm{d}u\bigg)\bigg|\tau_0 = t\bigg].$$

Given $\tau_0 = t$, the regular conditional $\mathbb{Q}_x$-distribution of $(X_{t-s}, 0 \leq s \leq t)$ is that of a 3-dimensional Bessel bridge from 0 to x over $[0, t]$ as a consequence of [19, Proposition VI.3.10 and Proposition VII.4.8]. On the canonical space $(C([0,1], \mathbb{R}^3), \mathbb{P}_{\mathsf{BB}^3})$ with coordinate process β, the process $\{|(s/t)x\mathrm{e}_1 + \sqrt{t}\beta_{s/t}|, 0 \leq s \leq t\}$ has the exact law of the aforementioned Bessel bridge. Therefore,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_x}\bigg[\exp\bigg(-\int_0^{\tau_0}\gamma(X_u)\mathrm{d}u\bigg)\bigg|\tau_0 = t\bigg] &= \mathbb{E}_{\mathsf{BB}^3}\bigg[\exp\bigg(-\int_0^t \gamma\bigg(\bigg|\frac{s}{t}x\mathrm{e}_1 + \sqrt{t}\beta_{s/t}\bigg|\bigg)\mathrm{d}s\bigg)\bigg] \\ &= \mathbb{E}_{\mathsf{BB}^3}\bigg[\exp\bigg(-t\int_0^1 \gamma\bigg(\bigg|ux\mathrm{e}_1 + \sqrt{t}\beta_u\bigg|\bigg)\mathrm{d}u\bigg)\bigg]; \quad (t, x) \in (\mathbb{R}_+)^2. \end{aligned}$$

Combining this with (A.4), the proof of Proposition 1.3 is complete.

A.2. **Proof of Proposition 2.2.** The following technical result is the backbone of the proof.

Lemma A.1. *Suppose that Assumption 2.1 holds, and define*

$$(A.5) \qquad \xi(t) := \exp(-tI(t)) := \exp\left(-t\int_0^1 \gamma\left(\left|uxe_1 + \sqrt{t}\beta_u\right|\right)\mathrm{d}u\right), \quad t \in \mathbb{R}_+.$$

Then, we have

$$|\xi(t) - \xi(s)| \le \Phi_T |t - s| \ \text{ for all } s \in [0, T], \text{ and } t \in [0, T],$$

where $\mathbb{E}_{\mathsf{BB}^3}[|\Phi_T|^m] < \infty$ *for all* $T \in \mathbb{R}_+$ *and* $m \in \mathbb{N}$.

Proof. First, note that (2.2) in Assumption 2.1 implies that for every $\kappa > 0$,

$$|a'(v_1)| \le |a'(0)| + |a'(v_1) - a'(0)| \le |a'(0)| + c_1 v_1 + c_2 \kappa^n v_1; \quad 0 \le v_1 \le \kappa,$$

$$|a(\kappa)| = \left|a(0) + \int_0^\kappa a'(u)\mathrm{d}u\right| \le |a(0)| + \int_0^\kappa |a'(u)|\mathrm{d}u \le |a(0)| + |a'(0)|\kappa + c_1\kappa^2 + c_2\kappa^{n+2},$$

$$|a(v_1) - a(v_2)| \le \left|\int_{v_1}^{v_2} a'(u)\mathrm{d}u\right| \le \left(|a'(0)| + c_1\kappa + c_2\kappa^{n+1}\right)|v_2 - v_1|; \quad 0 \le v_1 \le v_2 \le \kappa.$$

Using these inequalities, we obtain estimates for γ for every $\kappa > 0$ and every $0 \le v_1 \le v_2 \le \kappa$,

$$|\gamma(v_1)| = \left|\frac{a^2(v_1) + a'(v_1)}{2}\right| \le \frac{1}{2}\left(|a(0)| + |a'(0)|\kappa + c_1\kappa^2 + c_2\kappa^{n+2}\right)^2 + \frac{1}{2}\left(|a'(0)| + c_1\kappa + c_2\kappa^{n+1}\right) =: \varphi_1(\kappa),$$

$$\begin{aligned}
|\gamma(v_1) - \gamma(v_2)| &\le \frac{1}{2}\left|a^2(v_1) - a^2(v_2)\right| + \frac{1}{2}\left|a'(v_1) - a'(v_2)\right| \\
(A.6) \qquad &\le \frac{1}{2}|a(v_1) + a(v_2)||a(v_1) - a(v_2)| + \frac{1}{2}(c_1 + c_2\kappa^n)|v_1 - v_2| \\
&\le \left[\left(|a(0)| + |a'(0)|\kappa + c_1\kappa^2 + c_2\kappa^{n+2}\right)\left(|a'(0)| + c_1\kappa + c_2\kappa^{n+1}\right) + \frac{1}{2}(c_1 + c_2\kappa^n)\right]|v_1 - v_2| \\
&=: \varphi_2(\kappa)|v_1 - v_2|,
\end{aligned}$$

where $\mathbb{R}_+ \ni \kappa \mapsto \varphi_j(\kappa)$, $j = 1, 2$, are polynomial functions of $\kappa \in \mathbb{R}_+$ and do not depend on v_1, v_2.

Fix $T \in \mathbb{R}_+$. For $s \in [0, T]$, $t \in [0, T]$ and $u \in [0, 1]$, consider the random variables $\kappa = \sqrt{T}\max_{0 \le u \le 1}|\beta_u| + x$, $v(s, u) = |uxe_1 + \sqrt{s}\beta_u|$, $v(t, u) = |uxe_1 + \sqrt{t}\beta_u|$. Using the estimates established before, we obtain estimates for $I(t)$ in (A.5):

$$|I(t)| = \left|\int_0^1 \gamma(|uxe_1 + \sqrt{t}\beta_u|)\mathrm{d}u\right| \le \int_0^1 |\gamma(|uxe_1 + \sqrt{t}\beta_u|)|\mathrm{d}u = \int_0^1 |\gamma(v(t, u))|\mathrm{d}u \le \varphi_1(\kappa),$$

$$\begin{aligned}
s\left|I^1(t) - I^1(s)\right| &\le s\int_0^1 |\gamma(v(t, u)) - \gamma(v(s, u))|\mathrm{d}u \le s\int_0^1 |v(t, u) - v(s, u)|\varphi_2(\kappa)\mathrm{d}u \\
(A.7) \qquad &\le s(\sqrt{t} - \sqrt{s})\int_0^1 |\beta_u|\varphi_2(\kappa)\mathrm{d}u \le \frac{s(t - s)}{\sqrt{t} + \sqrt{s}}\frac{\kappa}{\sqrt{T}}\varphi_2(\kappa) \\
&\le \frac{s\kappa\varphi_2(\kappa)}{2\sqrt{sT}}(t - s) \le \frac{\kappa\varphi_2(\kappa)}{2}(t - s),
\end{aligned}$$

where we have used (A.6) in the second inequality, since $0 \le v(s, u) \le \kappa$, $0 \le v(t, u) \le \kappa$, used $|v(t, u) - v(s, u)| \le (\sqrt{t} - \sqrt{s})|\beta_u|$ in the third inequality, and $\max_{0 \le u \le 1} |\beta_u| \le \kappa/\sqrt{T}$ in the fourth inequality for $0 < s < t \le T$.

Finally, since γ is bounded from below by Assumption 2.1, so is I in (A.5), that is, $I^1(t) \ge \inf_{z \in \mathbb{R}_+} \gamma(z) > -\infty$ for every $t \ge 0$. With this observation, because of monotonicity and differentiability of exponential function, we obtain for $0 \le s < t \le T$,

$$|\xi(t) - \xi(s)| = \left| e^{-tI(t)} - e^{-sI(s)} \right| \le \left(e^{-T \inf \gamma} \vee 1 \right) |tI(t) - sI(s)|$$

$$= c_3 |(t - s)I(t) + s(I(t) - I(s))| \le c_3 |t - s| |I(t)| + c_3 s |I(t) - I(s)|,$$

where $c_3 := \exp(-T \inf \gamma) \vee 1 < \infty$. Combining this with the estimates for $|I(t)|$ and $s|I(t) - I(s)|$ in (A.7), we obtain, for $0 \le s < t \le T$,

$$|\xi(t) - \xi(s)| \le c_3 \left(\varphi_1(\kappa) + \frac{\kappa \varphi_2(\kappa)}{2} |t - s| \right) =: \varphi_3(\kappa) |t - s|,$$

where φ_1 and φ_2 are defined in (A.6), and hence φ_3 can be written as a polynomial function of κ whose coefficients do not depend on s nor on t but on T. Letting Φ_T be $\varphi_3(\kappa)$, and noting that all positive integer moments of maximum of standard 3-dimensional Bessel Bridge are finite ([18, Corollary 7]), we conclude the proof of Lemma A.1. $\qquad \square$

Let us define $\nu(t) := \mathbb{E}_{\mathsf{BB}^3}[\xi(t)]$ for $0 \le t \le T$. It follows from Lemma A.1 that ξ is locally Lipschitz continuous and moreover,

$$\mathbb{E}_{\mathsf{BB}^3} \left[|\xi(t) - \xi(s) - (\nu(t) - \nu(s))|^2 \right] \le c_4 |t - s|^2; \quad 0 \le s < t \le T.$$

Since the random paths $\{\widehat{\beta}^i, i = 1, \ldots, N\}$ are independent and identically distributed, for any $s, t \le T$ we obtain

(A.8)
$$\mathbb{E}_{\mathsf{BB}^3} \left| \eta^N(t) - \eta^N(s) \right|^2$$
$$= \left(q_x(t) \right)^2 \exp \left(-2 \int_0^x a(v) \mathrm{d}v \right) \mathbb{E}_{\mathsf{BB}^3} \left[|\xi(t) - \xi(s) - (\nu(t) - \nu(s))|^2 \right]$$
$$\le \left(\max_{0 \le t \le T} q_x(t) \right)^2 \exp \left(-2 \int_0^x a(v) \mathrm{d}v \right) c_4 |t - s|^2 =: c_5 |t - s|^2,$$

where the constant c_5 depends on T, x but not on N, s, t. This inequality is a sufficient condition for the tightness of the sequence $\{\eta^N \mid N \in \mathbb{N}\}$ of continuous stochastic processes starting at 0 in $C(\mathbb{R}_+, \mathbb{R})$ — see [11, Problem 2.4.12]. By the usual multi-dimensional central limit theorem, for each $n \ge 1$, $0 \le t_1 < \cdots < t_n < \infty$ the sequence $\left\{ \left(\eta^N(t_1), \eta^N(t_2), \ldots, \eta^N(t_n) \right) \mid N \in \mathbb{N} \right\}$ of random vectors converges in distribution to a Gaussian random vector with mean zero and and variance-covariance matrix $(\Gamma(t_i, t_j))_{1 \le i, j \le n}$, where

$$\Gamma(s, t) = \exp \left(-2 \int_0^x a(v) \mathrm{d}v \right) \mathrm{cov}_{\mathsf{BB}^3}[\exp(-sI(s)), \exp(-tI(t))], \quad (s, t) \in \mathbb{R}_+^2.$$

Therefore, we conclude that the tight sequence $\left\{\eta^N \mid N \in \mathbb{N}\right\}$ converges weakly to a continuous Gaussian process with mean zero and continuous covariance function Γ.

A.3. **Proof of Proposition 2.3.** Define that Gaussian tail function $\bar{\Phi}$ via

$$\bar{\Phi}(z) = \int_z^\infty \frac{e^{-y^2/2}}{\sqrt{2\pi}} \mathrm{d}y, \text{ for } z \in \mathbb{R}.$$

Furthermore, for fixed $T \in \mathbb{R}_+$, define the modulus of continuity in $\mathbb{L}^2$:

$$(A.9) \qquad \psi_T(h) := \max_{(s,t)\in[0,T]^2, |t-s|\leq h} \left[\mathbb{E}_{\mathsf{BB}^3}(\eta(t) - \eta(s))^2\right]^{1/2}, \text{ for } h \in [0,T].$$

It follows from (A.8) that $\psi_T(h) \leq \sqrt{c_5} h$ for $h \in \mathbb{R}_+$; therefore, $\int_1^\infty \psi_T(e^{-y^2})\mathrm{d}y < \infty$. We recall the following Fernique's inequality for Gaussian processes, which we shall use.

Lemma A.2 (Fernique's inequality — see (2.2) of [4])**.** *If the function ψ_T in (A.9) satisfies $\int_1^\infty \psi_T(e^{-y^2})\mathrm{d}y$, then for that fixed $T > 0$ and any integer $m \geq 2$,*

$$\mathbb{P}_{\mathsf{BB}^3}\left[\max_{0\leq t\leq T} |\eta(t)| > C_1(T,m)z\right] \leq C_2(m)\bar{\Phi}(z), \quad \text{for all } z > (1 + 4\log m)^{1/2},$$

where $C_1(T,m) := \max_{(s,t)\in[0,T]^2} \Gamma(s,t)^{1/2} + (2 + \sqrt{2})\int_1^\infty \psi_T(Tm^{-y^2})\mathrm{d}y$, and $C_2 := 5m^2\sqrt{2\pi}/2$.

The weak convergence of $\left\{\eta^N \mid N \in \mathbb{N}\right\}$ to η and the invariance principle for the maximum function imply that the sequence of normalized maxima $\left(\sqrt{N}\max_{t\in[0,T]} |\widehat{p}_x^N(t) - p_x(t)|\right)_{N\in\mathbb{N}}$ converges weakly to $\max_{0\leq t\leq T}\eta(t)$, where η is the limiting Gaussian process, as N goes to infinity. Since the law of the last random variable does not charge ∞, we conclude that the family $\left\{\sqrt{N}\max_{t\in[0,T]} |\widehat{p}_x^N(t) - p_x(t)| \mid N \in \mathbb{N}\right\}$ is bounded in probability.

A.4. **Proof of Proposition 3.1.** For the lower bound in (3.2) let us observe

$$\log \mathbb{E}_{\mathsf{BB}^3}\left[\exp\left(-t\int_0^1 \gamma\left(\left|u x \mathbf{e}_1 + \sqrt{t}\beta_u\right|\right)\mathrm{d}u\right)\right] \leq -t\inf_{z\in\mathbb{R}_+}\gamma(z); \quad t \in \mathbb{R}_+,$$

by Assumption 2.1. Therefore,

$$\inf_{z\in\mathbb{R}_+}\gamma(z) \leq \inf_{t\in\mathbb{R}_+}\left[-\frac{1}{t}\log\mathbb{E}_{\mathsf{BB}^3}\left[\exp\left(-t\int_0^1 \gamma\left(\left|u x \mathbf{e}_1 + \sqrt{t}\beta_u\right|\right)\mathrm{d}u\right)\right]\right] = \inf_{t\in\mathbb{R}_+}\lambda_x(t).$$

For the upper bound in (3.2), for a fixed $\kappa > 0$ let us consider $A_\kappa := \left\{\sqrt{t}\max_{0\leq u\leq 1}|\beta_u| \leq \kappa\right\}$. On A_κ, $|u x \mathbf{e}_1 + \sqrt{t}\beta_u| \leq \kappa + x$ holds for $0 \leq u \leq 1$; hence,

$$\int_0^1 \gamma\left(\left|u x \mathbf{e}_1 + \sqrt{t}\beta_u\right|\right)\mathrm{d}u \leq \max_{0\leq z\leq\kappa+x}\gamma(z) = m(\kappa + x),$$

where $m(w) := \max_{0\leq z\leq w}\gamma(z)$ for $w > 0$. It follows that, for $t \in \mathbb{R}_+$,

$$\mathbb{E}_{\mathsf{BB}^3}\left[\exp\left(-t\int_0^1 \gamma\left(\left|u x \mathbf{e}_1 + \sqrt{t}\beta_u\right|\right)\mathrm{d}u\right)\right] \geq \mathbb{E}_{\mathsf{BB}^3}\left[\exp\left(-t\int_0^1 \gamma\left(\left|u x \mathbf{e}_1 + \sqrt{t}\beta_u\right|\right)\mathrm{d}u\right)\mathbb{I}_{A_\kappa}\right]$$

$$\geq \exp\left(-tm(\kappa + x)\right)\mathbb{P}_{\mathsf{BB}^3}\left[\sqrt{t}\max_{0\leq u\leq 1}|\beta_u| \leq \kappa\right],$$

and hence

$$(A.10) \qquad \lambda_x(t) \leq m(\kappa + x) - \frac{1}{t} \log \mathbb{P}_{\mathsf{BB}^3} \left[\max_{0 \leq u \leq 1} |\beta_u| \leq \kappa t^{-1/2} \right].$$

The distribution for the maximum of the absolute value of the standard 3-dimensional Bessel bridge $|\beta|$ is known — see, for example, [18, (5)]. More precisely, we have

$$(A.11) \quad \mathbb{P}_{\mathsf{BB}^3} \left[\max_{0 \leq u \leq 1} |\beta_u| \leq \kappa t^{-1/2} \right] = \frac{2}{\kappa^3} \sqrt{\frac{2t}{\pi}} \sum_{n=1}^{\infty} \frac{n\pi}{J_{3/2}^2(n\pi)} \exp\left(-\frac{\pi^2 n^2}{2\kappa^2}t\right), \text{ for } \kappa > 0 \text{ and } t \in \mathbb{R}_+,$$

where $J_{3/2}$ is the Bessel function of index $3/2$. In particular,

$$\lim_{t \to \infty} \left(-\frac{1}{t} \mathbb{P}_{\mathsf{BB}^3} \log \left[\max_{0 \leq u \leq 1} |\beta_u| \leq \kappa t^{-1/2} \right] \right) = \frac{\pi^2}{2\kappa^2},$$

as only the first term in the summand in the series in (A.11) will play a role in the limit. Combining the last limiting relationship with inequality (A.10), we obtain

$$\limsup_{t \to \infty} \lambda_x(t) \leq m(\kappa + x) + \frac{\pi^2}{2\kappa^2}$$

Upon minimizing the right-hand side of the last inequality, the upper bound in (3.2) is obtained.

Finally, to verify (3.3), observe that γ being bounded from below implies that the random variables $\exp(-t \int_0^1 \gamma(|ux\mathbf{e}_1 + \sqrt{t}\beta_u|)\mathrm{d}u)$ are uniformly bounded for small $t \in \mathbb{R}_+$. Then, de L'Hôspital's rule and the bounded convergence theorem give

$$\lim_{t \downarrow 0} \lambda_x(t) = (-1) \lim_{t \downarrow 0} \frac{(\partial/\partial t)\,\mathbb{E}_{\mathsf{BB}^3}[\exp(-t \int_0^1 \gamma(|ux\mathbf{e}_1 + \sqrt{t}\beta_u|)\mathrm{d}u)]}{\mathbb{E}_{\mathsf{BB}^3}[\exp(-t \int_0^1 \gamma(|ux\mathbf{e}_1 + \sqrt{t}\beta_u|)\mathrm{d}u)]} = \int_0^1 \gamma(ux)\mathrm{d}u = \frac{1}{x}\int_0^x \gamma(u)\mathrm{d}u.$$

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(Tomoyuki Ichiba) Department of Statistics and Applied Probability, University of California Santa Barbara, CA 93106, South Hall 5607A

  E-mail address: ichiba@pstat.ucsb.edu

(Constantinos Kardaras) Department of Mathematics and Statistics, Boston University, 111 Cummington Street, Boston, MA 02215, USA

  E-mail address: kardaras@bu.edu